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Linear Systems with Transfer Functions of Bounded Type : Canonical Factorization *

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ABSTRACT

This paper deals with linear discrete-time systems with matrix-valued transfer functions each entry of which is represented as the quotient of two H^{∞}-class functions. The notion of outer functions (or functions of minimum phase) is extended to matrix-valued functions of the Nevanlinna class N, and a canonical factorization theorem for matrix functions of class N is presented. This theorem gives minimum phase systems for these linear systems, and specifies a necessary and sufficient condition for the systems to be causal. The notion of the matrix-fraction descriptions (MFDs) is extended to these systems, and some properties of the MFDs are presented by means of the canonical factorization theorem.

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I. INTRODUCTION

In the design of controllers for linear systems (or plants), the systems are usually described by rational transfer functions. In the analysis of time series or digital signals, however, they could not be assumed a priori to be generated from linear systems with rational transfer functions. It is more natural to assume that they have irrational transfer functions. The theory of linear systems with rational transfer functions should be extended to the irrational case in order to cover the field of time series or digital signal analysis.

Dewilde deals with a class of linear continuous-time systems with irrational transfer functions, which he called roomy systems [13]. The roomy systems exhibits some energy conservation properties. The systems considered in this paper have no restriction of energy conservation, and thus contains the discrete-time version of the roomy systems.

In this paper we shall deal with linear discrete-time systems with matrixed-valued transfer functions each entry of which is represented as the quotient of two bounded functions analytic in the unit disk $|z| < 1$[1]. In the literature of mathematics, such a function is called of bounded type. That is, a scalar function $h(z)$ of bounded type is written in the form $h(z) = n(z) / d(z)$, where $n(z)$ and $d(z)$ belong to the Hardy class H^{∞}(that is, the class of functions bounded in $|z| < 1$).

First, we shall take out several properties of scalar analytic functions of the Hardy class $H^p (0 < p \leqq \infty)$ and the Nevanlinna class [1-5] which are

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[1] It is a common practice in engineering to interpret the argument z^{-1} in the z-transfer functions as a unit-delay operator. It is, however, convenient in the sequel to interpret the argument z as a unit-delay operator in order to exploit the Hardy space theory.

nesessary for our purpose. The notion of outer functions (or functions of minimum phase) is extended to matrixed-valued functions of the Nevanlinna class N. We shall next provide a canonical factorization theorem for matrix functions of class N. This theorem gives linear systems of minimum phase corresponding to linear systems with transfer functions of bounded type, and specifies a necessary and sufficient condition for the systems to be causal.

The notion of the matrix-fraction description (MFDs) is extended to these systems, and some properties of the MFDs are presented by means of the canonical factorization theorem.

The following notation will be used in this paper. The symbols, rank H, det H, adj H and H^* denote, respectively, the rank, the determinant, the adjoint matrix and the conjugate transpose of a matrix H. For Hermitian matrix K and L, $K \geqq L$ means $K - L$ is positive semidefinite. I_r denotes the $r \times r$ identity matrix, and its subscript r is omitted when it is clear from the context.

II. MATHEMATICAL PRELIMINARIES FOR ANALYTIC FUNCTIONS

This section contains some basic results on scalar analytic functions of the Hardy class $H^p (0 < p \leqq \infty)$ and the Nevanlinna class N, and on matrix-valued functions of the Hardy class H^2, which are necessary for our purpose. These are presented without proofs. The reader is referred to the textbooks [1], [2] for an introduction to the Hardy and the Nevanlinna spaces, and the textbooks [3], [4] for the results related to functions of bounded type. He is also referred to the works [6-11] for properties on matrix-valued functions of the Hardy class H^2.

A) Scalar functions of the Nevanlinna class N.

We shall set down the definitions of these subclasses in the class of analytic functions in the unit disk $|z| < 1$.

A scalar functions $f(z)$ analytic in $|z| < 1$ is said to be of class H^p ($0 < p \leqq \infty$) if the integral means

$$\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{j\omega})|^p d\omega\}^{1/p}, \quad \text{for } 0 < p < \infty; \tag{1a}$$

$$\max_{\omega}|f(re^{j\omega})| \quad , \quad \text{for } p = \infty \tag{1b}$$

are bounded for $r < 1$. Thus H^∞ is the class of bounded function analytic in the unit disk, while H^2 is the class of power series $\sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. A scalar function $f(z)$ analytic in $|z| < 1$ is said to be of class N if the characteristic function $T(r)$ defined by

$$T(r) = \int_{-\pi}^{\pi} \log^+|f(re^{j\omega})|d\omega \tag{2}$$

is bounded for all $r < 1$, where $\log^+ x = \log x$ if $x \geqq 1$ and $\log^+ x = 0$ if $0 \leqq x < 1$. It is clear that N contains H^p for every $p > 0$. When $f(z)$ is analytic in $|z| < 1$, the integral means (1) and the characteristic function $T(r)$ are nondecreasing as r increases. A scalar measurable function $f(e^{j\omega})$ on the unit circle $|e^{j\omega}| = 1$ is said to belong to class $L^p (0 < p \leqq \infty)$ if the integral means

$$\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{j\omega})|^p d\omega\}^{1/p}, \quad \text{for } 0 < p < \infty; \tag{3a}$$

$$\text{ess.}\sup_{\omega}|f(e^{j\omega})| \quad , \quad \text{for } p = \infty \tag{3b}$$

are finite. A matrix valued function F is called, respectively, of calss

H^p, of class N and of class L^p if its every entry F_{ij} belongs to class H^p, class N and class L^p.

The following theorem is a fundamental result for functions of bounded type [3, p.188; 4, p.157].

Theorem 1 (R. Nevanlinna): A function f(z) meromorphic in $|z| < 1$ is of bounded type if and only if the characteristic function T(r) defined by (2) is bounded as $r \to 1$. Thus, an analytic function in $|z| < 1$ belongs to the class N if and only if it is of bounded type.

Since bounded analytic function f(z) has almost everywhere (a.e.) the boundary value $f(e^{j\omega})$ which is defined by the radial limit

$$f(e^{j\omega}) = \lim_{r\to 1} f(re^{j\omega}) \quad \text{a.e.},$$

a function f(z) of bounded type also has a.e. the boundary value.

In order to show the factorization theorems on functions of the Hardy and the Nevanlinna classes, we shall write down the definitions of outer functions and inner functions.

An outer function for the class N is a function of the form

$$f(z) = c \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{j\omega}+z}{e^{j\omega}-z} \log \psi(e^{j\omega})d\omega\}, \qquad |z| < 1 \tag{4}$$

where c is a complex number such that $|c| = 1$, $\psi(e^{j\omega}) \geqq 0$ and $\log \psi(e^{j\omega}) \in L^1$. We note in (4) that it holds

$$\lim_{r\to 1}|f(re^{j\omega})| = \psi(e^{j\omega}) \quad \text{a.e.}$$

An inner function is a function $f \in H^\infty$ for which $|f(e^{j\omega})| = 1$ a.e. There are two kinds of inner functions. The one is a Blaschke product B(z) which is defined by

$$B(z) = z^m \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \tag{5}$$

where m is a nonnegative integer and $\sum(1 - |a_n|) < \infty$. The set $\{a_n\}$ may be finite, or even empty. The other is a singular inner function S(z) which is defined by

$$S(z) = \exp\{-\int_{-\pi}^{\pi} \frac{e^{j\omega} + z}{e^{j\omega} - z} d\mu(\omega)\} \tag{6}$$

where $\mu(\omega)$ is a bounded nondecreasing function such that the derivative $\mu'(\omega) = 0$ a.e.

The following theorem is a generalization of the factorization theorem by F. Riesz.

<u>Theorem 2 (Smirnov's canonical factorization theorem)</u>: Every function $f(z) \not\equiv 0$ of class N can be expressed in the form

$$f(z) = B(z)\{S_1(z)/S_2(z)\}f_o(z) \tag{7}$$

where B(z) is a Blaschke product, $S_1(z)$ ans $S_2(z)$ are singular inner functions, and $f_o(z)$ is an outer functions (with $\psi(e^{j\omega}) = f(e^{j\omega})$). Conversely, every function of the form (7) belongs to N. Moreover, every function f(z) $\not\equiv 0$ belongs to H^p if and only if it can be expressed in the form (7), where $S_2(z) = 1$ and $f(e^{j\omega}) \in L^p$.

The above form (7) for H^p is called the <u>inner-outer factorization</u> for class H^p.

Another subclass in class N was introduced by Smirnov. A function f(z) is called of class N^+(or called "D" by Smirnov) if it has the form (7) such that $S_2(z) = 1$, that is, the form $f(z) = B(z)S(z)f_o(z)$, where B(z) is a

Blaschke product, $S(z)$ is a singular inner function and $f_o(z)$ is an outer function. For a matrix function $F(z)$, it is said to be of class N^+ if its every entry belongs to class N^+. It is clear by the definitions that an outer function for class N belongs to class N^+.

The following useful result is an easy consequence of the canonical factorization theorem.

Theorem 3 (Smirnov): If $f \in N^+$ and $f(e^{j\omega}) \in L^p$ for some p, then $f \in H^p$.

The following theorem characterizes outer functions, which is derived easily by the canonical factorization theorem.

Theorem 4: The following properties are equivalent.

1) $f(z)$ is outer for class N.

2) $f(z) \not\equiv 0$ belongs to class N and satisfies

$$|f(0)| \geqq |\tilde{f}(0)| \tag{8a}$$

for any function $\tilde{f}(z)$ of class N^+ with the same magnitude of the boundary value as $f(z)$, that is,

$$|\tilde{f}(e^{j\omega})| = |f(e^{j\omega})| \quad \text{a.e.} \tag{8b}$$

3) $f(z)$ belongs to class N and

$$|f(0)| = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{j\omega})|d\omega > 0. \tag{9}$$

B) Matrix-valued functions of the Hardy class H^2.

A square matrix function $U(z)$ is called inner if $U(z)$ is of class H^∞ and if $U(e^{j\omega})$ is unitary a.e., that is, $U(e^{j\omega})U(e^{j\omega})^* = I$ a.e.

The definition of outer matrix functions was first introduced by Helson and Lowdenslager (1961) for the Hardy class H^2[11]. This is shown to be equivalent to that of maximal factors introduced by Krein (1958) and Rozanov (1959) [6], [7], and to that of optimal factors by Masani (1962) [9], in the investigations of weekly stationary random processes. The definition by Helson and Lowdenslager can not be applied to class N. The definition by Krein and others, however, can be exploited to class N. Thus, the notion of outer functions will be extended for the first time to matrix functions of class N.

An $r \times m$ matrix function $F(z)$ of class N is said to be column outer if $F(e^{j\omega})$ has a.e. full column rank[2], that is,

$$\operatorname{rank} F(e^{j\omega}) = m \quad \text{a.e.,} \tag{10}$$

and if

$$F(0)F(0)^* \geqq \tilde{F}(0)\tilde{F}(0)^* \tag{11a}$$

for any $r \times m$ matrix function $\tilde{F}(z)$ of class N^+ whose boundary value $\tilde{F}(e^{j\omega})$ satisfies

$$\tilde{F}(e^{j\omega})\tilde{F}(e^{j\omega})^* = F(e^{j\omega})F(e^{j\omega})^* \quad \text{a.e.} \tag{11b}$$

A row outer matrix function is defined in the same way, replacing respectively (10) by

$$\operatorname{rank} F(e^{j\omega}) = r \quad \text{a.e.,} \tag{12}$$

(11a) by

$$F(0)^*F(0) \geqq \tilde{F}(0)^*\tilde{F}(0), \tag{13a}$$

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2 The notion of column outer functions may be valid without this assumption as in [11], but unnecessary details shoud be taken into account in this case. Thus we have the assumption (10) in this paper.

and (11b) by

$$\tilde{F}(e^{j\omega})^*\tilde{F}(e^{j\omega}) = F(e^{j\omega})^*F(e^{j\omega}) \quad \text{a.e.} \tag{13b}$$

For a matrix function F(z) analytic in $|z| < 1$ and having the form

$$F(z) = \sum_{n=0}^{\infty} F_n z^n, \quad |z| < 1,$$

define the analytic function $F_*(z)$ by

$$F_*(z) = F(1/\bar{z})^* = \sum_{n=0}^{\infty} F_n^* z^n, \quad |z| < 1.$$

Then it is clear that $F \varepsilon N$ is row outer if and only if $F_* \varepsilon N$ is column outer.

The following theorem provides the inner-outer factorization for class H^2.

Theorem 5 (Krein and Rozanov): Every $r \times m$ matrix function $F \varepsilon H^2$ with rank $F(e^{j\omega}) = m$ a.e. can be expressed in the form

$$F(z) = F_o(z)F_i(z) \tag{14}$$

where $F_i(z)$ is inner, and $F_o(z) \varepsilon H^2$ is column outer. Moreover, the factorization (14) is unique up to multiplication by a constant unitary matrix, that is if $F(z) = \tilde{F}_o(z)\tilde{F}_i(z)$ is another inner-outer factorization, then there exists a constant unitary matrix U such that

$$\tilde{F}_o(z) = F_o(z)U, \quad \tilde{F}_i(z) = U^*F_i(z). \tag{15}$$

The following theorem characterizes a square matrix outer function of class H^2.

Theorem 6 (Wiener and Masani): Suppose F(z) is a square matrix function of class H^2. Then $F(z)$ is column outer (or row outer) if and only if

$$|\det F(0)| = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\det F(e^{j\omega})|d\omega\} > 0. \tag{16}$$

It is clear by Theorems 4 and 6 that a square matrix function $F(z)$ of class H^2 is outer if and only if $\det F(z)$ is a scalar outer function. It is shown by this statement and Theorem 5 that a column outer function $F(z)$ of class H^2 has full column rank for all $|z| < 1$ [7. p.83].

III. CANONICAL FACTORIZATION FOR MATRIX FUNCTIONS OF CLASS N

In this section the canonical factorization theorem in the scalar case will be extended to the case of matrix functions of class N, and the inner-outer factorization of matrix functions of class N^+ will be presented, which is a generalization of Theorem 5.

We first introduce some notions about greatest common inner divisors of H^∞-class matrix functions. Let $A \varepsilon H^\infty$ be $r\times\ell$ dimensional, and $B \varepsilon H^\infty$ be $r\times m$ dimensional. We say that A and B have a common left inner divisor (CLID) if there exist an $r \times r$ matrix inner function U such that

$$A = UA_1, \quad B = UB_1, \quad A_1 \varepsilon H^\infty, \quad \text{and } B_1 \varepsilon H^\infty.$$

U is called a greatest common left inner divisor (GCLID) if, for any CLID U_1, there exists an inner function U_2 such that

$$U = U_1U_2.$$

In the above definitions we tacitly assume that matrix $[A(z), B(z)]$ have full row rank a.e. in $|z| \leqq 1$. It is clear by Theorem 5, since $H^\infty \subset H^2$, that a

GCLID U is unique except for a constant right unitary factor. A and B are said to be <u>left inner coprime</u> (LIC) if their GCLID is a constant unitary matrix.

Using Theorem 5 (or the inner-outer factorization for class H^∞), we easily obtain the following result.

<u>Theorem 7</u>: Let $[A, B] \varepsilon H^\infty$ have a.e. full row rank. Then A and B are LIC if and only if [A, B] is row outer.

We shall start with an $r \times m$ matrix function H(z) of bounded type. Since each entry of H(z) can be represented as a quotient of two bounded functions, we can write it as

$$H(z) = \frac{N(z)}{d(z)} \tag{17}$$

where d(z) is the product of all denominators of the entry of H(z), which is a scalar function of class H^∞. Then $N(z) = d(z)H(z)$ is an $r \times m$ matrix function of class H^∞. Thus we can represent H(z) as a <u>left matrix-fraction description</u> (left MFD)

$$H(z) = D_L(z)^{-1}N_L(z), \tag{18a}$$

$$D_L(z) = d(z)I_r, \quad \text{and } N_L(z) = N(z). \tag{18b}$$

In general, $D_L(z)$ and $N_L(z)$ are not LIC, and thus there exists a GCLID U such that

$$[D_L(z), N_L(z)] = U(z)[\hat{D}(z), \hat{N}(z)] \tag{19}$$

where $[\hat{D}(z), \hat{N}(z)]$ is row outer by Theorem 7. A left MFD $[\hat{D}(z), \hat{N}(z)]$ is said to be <u>irreducible</u> if $\hat{D}(z)$ and $\hat{N}(z)$ are LIC.

We shall provide one of main results in this paper, which is an extension of the Smirnov canonical factorization theorem to the matrix case.

<u>Theorem 8 (Canonical factorization theorem):</u> Let $H(z)$ be of class N and $H(z) = D(z)^{-1}N(z)$ be an irreducible left MFD. Then $\det D(z)$ has no zero in $|z| < 1$. Let

$$\det D(z) = d_o(z)d_s(z), \tag{20a}$$

$$\operatorname{adj} D(z) = A_o(z)A_s(z), \tag{20b}$$

$$A_s(z)N(z) = L_o(z)L_i(z) \tag{20c}$$

be the inner-outer factorizations of $\det D(z)$, $\operatorname{adj} D(z)$ and $A_s(z)N(z)$, respectively, where $d_o(z)$ is outer, $d_s(z)$ is singular inner, $A_o(z)$ is outer, $A_s(z)$ is singular inner, $L_o(z)$ is column outer, and $L_i(z)$ is inner. Then $H(z)$ can be expressed in the form

$$H(z) = H_o(z)\cdot H_i(z)\cdot\frac{1}{d_s(z)} \tag{21a}$$

where $H_o(z)$ is column outer for class N, $H_i(z)$ is inner, and they are respectively defined by

$$H_o(z) = \frac{1}{d_o(z)} A_o(z)L_o(z), \tag{21b}$$

$$H_i(z) = L_i(z). \tag{21c}$$

Moreover, $H(z)$ belongs to class N^+ if and only if it can be expressed in the form (21) with $d_s(z) = 1$ and $A_s(z) = I$ (or $A_o(z) = \operatorname{adj} D(z)$). In this case, the inner-outer factorization (21a) is unique up to multiplication by a constant unitary matrix, that is, if $H(z) = \tilde{H}_o(z)\tilde{H}_i(z)$ is another inner-outer factorization, then there exists a constant unitary matrix U such that

$$\tilde{H}_o(z) = H_o(z)U \quad\text{and}\quad \tilde{H}_i(z) = U^*H_i(z). \tag{22}$$

Proof: Since $[D(z), N(z)]$ is row outer, using the comment below Theorem 6, we have

$$\text{rank } [D(z), N(z)] = r \quad \text{for all} \quad |z| < 1. \tag{23}$$

Substituting $N(z) = D(z)H(z)$ for $N(z)$ in (23) yields

$$[D(z), N(z)] = D(z)[I_r, H(z)]. \tag{24}$$

Since $H(z)$ is analytic in $|z| < 1$, (23) and (24) imply that $\det D(z) \not\equiv 0$ for all $|z| < 1$. Thus there is no Blaschke product in the inner-outer factorization of $\det D(z)$ as in (20a). Since $\text{adj } D(z)$ belongs to class H^∞, by means of Theorem 5, we obtain the inner-outer factorization (20b) of $\text{adj } D(z)$. We shall show below that the inner factor $A_s(z)$ becomes singular. The identity

$$D(z)^{-1} = \frac{1}{\det D(z)} \cdot \text{adj } D(z) \tag{25}$$

gives

$$D(z) \cdot \text{adj } D(z) = \det D(z) \cdot I_r,$$

which yields together with (20a)

$$\det[\text{adj } D(z)] = (\det D(z))^{r-1} = d_s(z)^{r-1} \cdot d_o(z)^{r-1}. \tag{26}$$

Thus it follows from (20b) and (26)

$$\det A_s(z) = c \cdot d_s(z)^{r-1}, \tag{27}$$

where $|c| = 1$. Thus the inner function $A_s(z)$ becomes singular. Since $A_s(z)N(z)$ belongs to class H^∞, we get the inner-outer factorization (20c) by means of Theorem 5. It is clear from (20), (21b), (21c) and (25) that $H(z)$ can be represented in the form (21a) provided that $H_o(z)$ given by (21b) is column outer. We shall show that $H_o(z)$ is a column outer function for class N. It is clear by Theorems 1 and 2 that $H_o(z)$ belongs to class N (in fact, class N^+ by Theorem 5). Consider any $r \times m$ matrix function $\bar{H} \in N^+$ such that

$$\bar{H}(e^{j\omega})\bar{H}(e^{j\omega})^* = H_o(e^{j\omega})H_o(e^{j\omega})^* \quad \text{a.e.} \tag{28}$$

It follows from (21b) and (28) that

$$A_o(e^{j\omega})^{-1}d_o(e^{j\omega})\bar{H}(e^{j\omega})\bar{H}(e^{j\omega})^*d_o(e^{j\omega})^*A_o(e^{j\omega})^{*-1}$$
$$= L_o(e^{j\omega})L_o(e^{j\omega})^* \quad \text{a.e.} \tag{29}$$

Since $A_o(z)$ is outer for class H^∞, Theorems 1, 2 and 5 imply that $A_o(z)^{-1}$ belongs to class N^+. Since $d_o \in H^\infty$ and $\bar{H} \in N^+$, this fact yields $A_o^{-1}\cdot d_o\cdot\bar{H} \in N^+$ (in fact, $A_o^{-1}\cdot d_o\cdot\bar{H} \in H^\infty$ by Theorem 3). By the fact that $L_o(z)$ is column outer, we have

$$A_o(0)^{-1}d_o(0)\bar{H}(0)\bar{H}(0)^*d_o(0)^*A_o(0)^{*-1}$$
$$\leqq L_o(0)L_o(0)^*. \tag{30}$$

This gives, together with (21b)

$$\bar{H}(0)\bar{H}(0)^* \leqq H_o(0)H_o(0)^*, \tag{31}$$

which means $H_o(z)$ is column outer.

Next suppose $H \in N^+$. Then we can show below that $D(z)$ is outer for class H^∞, which implies, by the coment below Theorem 6, that det $D(z)$ is outer, which yields from (26) that adj $D(z)$ is also outer. Thus we can take $d_s(z) = 1$ in (20a) and $A_s(z) = I$ in (20b). Consider any $r \times r$ matrix function $\bar{D} \in H^\infty$ such that

$$\bar{D}(e^{j\omega})^*\bar{D}(e^{j\omega}) = D(e^{j\omega})^*D(e^{j\omega}) \quad \text{a.e.} \tag{32}$$

Denote

$$\bar{N}(z) = \bar{D}(z)H(z). \tag{33}$$

By Theorem 2 we get $\bar{N} \varepsilon N^+$, because $\bar{D} \varepsilon H^\infty$ and $H \varepsilon N^+$. It follows from (32), (33) and $H(z) = D(z)^{-1}N(z)$ that

$$\bar{D}(e^{j\omega})^*\bar{N}(e^{j\omega}) = D(e^{j\omega})^*N(e^{j\omega}) \quad \text{a.e.} \tag{34a}$$

$$\bar{N}(e^{j\omega})^*\bar{N}(e^{j\omega}) = N(e^{j\omega})^*N(e^{j\omega}) \quad \text{a.e.} \tag{34b}$$

Since $\bar{N} \varepsilon N^+$ and $N \varepsilon H^\infty$, using Theorem 3 and (34b), we obtain $\bar{N} \varepsilon H^\infty$. (32) and (34) become

$$[\bar{D}(e^{j\omega}), \bar{N}(e^{j\omega})]^*[\bar{D}(e^{j\omega}), \bar{N}(e^{j\omega})]$$

$$= [D(e^{j\omega}), N(e^{j\omega})]^*[D(e^{j\omega}), N(e^{j\omega})] \quad \text{a.e.} \tag{35}$$

$[D(z), N(z)]$ is row outer for class H^∞ by Theorem 7, because $D(z)$ and $N(z)$ are LIC. This implies

$$[\bar{D}(0), \bar{N}(0)]^*[D(0), N(0)] \leqq [D(0), N(0)]^*[D(0), N(0)], \tag{36}$$

which gives

$$\bar{D}(0)^*\bar{D}(0) \leqq D(0)^*D(0) \tag{37}$$

Thus $D(z)$ becomes row outer (or outer since $D(z)$ is square).

Conversely, suppose $d_s(z) = 1$ in (21a). Since $H_o(z)$ given by (21b) belongs to class N^+ by Theorem 2, $H(z)$ also belongs to class N^+, because $H_i \varepsilon H^\infty$.

Although the proof of the fact that the inner-outer factorization of $H(z)$ of class N^+ is unique up to multiplication by a constant unitary matrix can be carried out in a similar way to the corresponding proof in Theorem 5, we shall present it for completeness. Suppose $H \varepsilon N^+$ and let

$$H(z) = \tilde{H}_o(z)\tilde{H}_i(z) \tag{38}$$

be another inner-outer factorization of $H(z)$, where $\tilde{H}_o(z)$ is column outer and $\tilde{H}_i(z)$ is inner. The identity

$$\tilde{H}_o(z)\tilde{H}_i(z) = H_o(z)H_i(z) \tag{39}$$

gives

$$\tilde{H}_o(z) = H_o(z)U(z), \tag{40a}$$

where

$$U(z) = H_i(z)\tilde{H}_i(z)^{-1}. \tag{40b}$$

Since $H_o(z)$ is column outer, $H_o(z)$ has full column rank for all $|z| < 1$ (see (21b) and the comment below Theorem 5). This, together with the fact that $\tilde{H}_o(z)$ is analytic in $|z| < 1$, means that the matrix $U(z)$ satisfying (40a) is analytic in $|z| < 1$. By the definition of inner functions, we have from (39)

$$\tilde{H}_o(e^{j\omega})H_o(e^{j\omega})^* = H_o(e^{j\omega})H_o(e^{j\omega})^* \quad \text{a.e.} \tag{41}$$

Since $\tilde{H}_o(z)$ and $H_o(z)$ are column outer, (41) implies

$$\tilde{H}_o(0)\tilde{H}_o(0)^* = H_o(0)H_o(0)^*. \tag{42}$$

Substituting (40a) for $\tilde{H}_o(e^{j\omega})$ in (41) yields

$$H_o(e^{j\omega})U(e^{j\omega})U(e^{j\omega})^* H_o(e^{j\omega})^* = H_o(e^{j\omega})H_o(e^{j\omega})^* \quad \text{a.e.}$$

which gives

$$U(e^{j\omega})U(e^{j\omega})^* = I \quad \text{a.e.,} \tag{43}$$

because $H_o(e^{j\omega})$ has a.e. full column rank. Similarly, substituting (40a) for $\tilde{H}_o(0)$ in (42) yields

$$U(0)U(0)^* = I. \tag{44}$$

By means of the maximum modulus theorem of analytic functions [2], (43) and (44) imply that $U(z)$ becomes a constant unitary matrix. Thus we obtain (22) from (40). Q.E.D.

In Theorem 8, if the given $H(z)$ is of bounded type, (20a), (20b) and (20c) must be exchanged respectively for the inner-outer factorizations

$$\det D(z) = d_o(z)d_i(z), \tag{20'a}$$

$$\operatorname{adj} D(z) = A_o(z)A_i(z), \tag{20'b}$$

$$A_i(z)N(z) = L_o(z)L_i(z), \tag{20'c}$$

where $d_o(z)$, $A_o(z)$ and $L_o(z)$ are column outer, and $d_i(z)$, $A_i(z)$ and $L_i(z)$ are inner. Then $H(z)$ can be represented in the form

$$H(z) = H_o(z)\cdot H_i(z)\cdot\frac{1}{d_i(z)}, \tag{21'a}$$

where $H_o(z)$ is column outer, $H_i(z)$ is inner and they are obtained by means of (20'), (21b) and (21c).

We note that $H_o(z)$ has full column rank for all $|z| < 1$, since $A_o(z)L_o(z)$ has full column rank for all $|z| < 1$.

The notions about the GLIDs introduced in the beginning of this section are extended to the class N^+. Then we obtain the same result for class N^+ as Theorem 7.

Theorem 9: Let $[A, B] \in N^+$ have a.e. full row rank. Then A and B are LIC if and only if [A, B] is row outer.

Using Theorems 8 and 9, we have basic properties of outer functions for class N. We shall state them for row outer functions for convenience of later use.

Theorem 10: Let H(z) be an $r \times m$ matrix function of class N.

1) Suppose that H(z) is square and belongs to class N^+. Then H(z) is row outer (or column outer) if and only if det H(z) is outer.
2) If H(z) is square and outer, then its inverse $H(z)^{-1}$ is also outer.
3) If H(z) has a factorization

$$H(z) = H_1(z)H_2(z), \quad H_1 \in N, \text{ and } H_2 \in N \tag{45}$$

   where $H_1(z)$ is square. Then H(z) is row outer if and only if both $H_1(z)$ and $H_2(z)$ are row outer.
4) Suppose $H \in N^+$. Then H(z) is row outer if and only if there exists an $r \times r$ submatrix of H(z) which is outer.
5) Suppose $H \in N^+$. Then H(z) is row outer if and only if there exists a right inverse G(z) of H(z) in class N^+.
6) Let $H(z) = H_i(z)H_o(z)$ be the inner-outer factorization of H(z), where $H_i(z)$ is inner and $H_o(z)$ is row outer. Then det $H_i(z)$ is a greatest common inner divisor of all the minors of order r in H(z).

Proof: Proof of 1) If H(z) is (column) outer, from (21b) and the comment for class H^2 below Theorem 6, det H(z) is scalar outer for class N. Let $H(z) = H_i(z)H_o(z)$ be the inner outer factorization of H(z), where $H_i(z)$ is inner and $H_o(z)$ is row outer. Then we have $\det H(z) = \det H_i(z) \cdot \det H_o(z)$. Suppose that $\det H(z)$ is not outer. Then $\det H_i(z)$ is not constant. Thus $H_i(z)$ is not a constant unitary matrix. Therefore $H(z)$ is not row outer.

<u>Proof of 2)</u> Since $H(z)$ is outer, $\det H(z)$ has no zero in $|z| < 1$. Thus by Theorem 1 $H(z)^{-1}$ belongs to class N. Consider any square matrix function $L \in N^+$ such that

$$L(e^{j\omega})^* L(e^{j\omega}) = H(e^{j\omega})^{-1*} H(e^{j\omega})^{-1} \quad \text{a.e.,} \tag{46}$$

which gives

$$H(e^{j\omega})^* L(e^{j\omega})^* L(e^{j\omega}) H(e^{j\omega}) = I \quad \text{a.e.}$$

Since the identity matrix is outer, this means

$$H(0)^* L(0)^* L(0) H(0) \leqq I.$$

Thus

$$L(0)^* L(0) \leqq H(0)^{-1*} H(0)^{-1},$$

which implies together with (46) that $H(z)^{-1}$ is outer.

<u>Proof of 3)</u> "if" part: Consider any square matrix function $\bar{H} \in N^+$ such that

$$\bar{H}(e^{j\omega})^* \bar{H}(e^{j\omega}) = H_1(e^{j\omega})^* H_1(e^{j\omega}) \quad \text{a.e.,} \tag{47}$$

which gives together with (45)

$$\begin{aligned} H_2(e^{j\omega})^* \bar{H}(e^{j\omega})^* \bar{H}(e^{j\omega}) H_2(e^{j\omega}) &= H_2(e^{j\omega})^* H_1(e^{j\omega})^* H_1(e^{j\omega}) H_2(e^{j\omega}) \\ &= H(e^{j\omega})^* H(e^{j\omega}) \quad \text{a.e.} \end{aligned}$$

Since H(z) is row outer, this implies

$$H_2(0)^*\bar{H}(0)^*\bar{H}(0)H_2(0) \leqq H(0)^*H(0)$$

$$= H_2(0)^*H_1(0)^*H_1(0)H_2(0).$$

Since H(z) has full row rank for all $|z| < 1$, $H_2(0)$ has full row rank from (45). Thus the above relation gives

$$\bar{H}(0)^*H(0) \leqq H_1(0)^*H_1(0). \tag{48}$$

Therefore, $H_1(z)$ is row outer. Let

$$H_2(z) = H_{2i}(z)H_{2o}(z), \tag{49}$$

be the inner-outer factorization of $H_2(z)$, where $H_{2i}(z)$ is inner and $H_{2o}(z)$ is row outer. Put

$$L(z) = H_1(z)H_{2i}(z). \tag{50}$$

Then we get the factorization

$$H(z) = L(z)H_{2o}(z).$$

Since L(z) is square, the above argument from (47) to (48) shows that L(z) is outer, which means that $H_{2i}(z)$ is a constant unitary matrix. Thus $H_2(z)$ is row outer.

"only if" part: Suppose that $H_1(z)$ and $H_2(z)$ are row outer. Then from (45) we have the factorization

$$H_2(z) = H_1(z)^{-1}H(z), \tag{51}$$

where $H_1^{-1} \varepsilon$ N by 2) of Theorem 1, because H(z) is outer. Since $H_2(z)$ is row outer, we know from the above "only if" part that $H(z)$ is row outer.

Proof of 4) Suppose $H(z)$ is row outer. Then, by the definition of row outer functions (see (12)), there exists an $r \times r$ submatrix $P_1(z)$ of $H(z)$ such that $P_1(e^{j\omega})$ has a.e. full rank. Interchanging some columns in $H(z)$ can place $P_1(z)$ in the left side of $H(z)$, which does not affect the fact that $H(z)$ is row outer. Thus we can take $H(z) = [P_1(z), P_2(z)]$ without loss of generality. Consider any square matrix function $\bar{P}_1 \in N^+$ such that

$$\bar{P}_1(e^{j\omega})^* \bar{P}_1(e^{j\omega}) = P_1(e^{j\omega})^* P_1(e^{j\omega}) \quad \text{a.e.} \tag{52}$$

put

$$\bar{P}_2(z) = \bar{P}_1(z) P_1(z)^{-1} P_2(z), \tag{53a}$$

$$\bar{H}(z) = [\bar{P}_1(z), \bar{P}_2(z)]. \tag{53b}$$

Then we have

$$\bar{H}(e^{j\omega})^* \bar{H}(e^{j\omega}) = H(e^{j\omega})^* H(e^{j\omega}).$$

Since $H(z)$ is row outer, this implies

$$\bar{H}(0)^* \bar{H}(0) \leqq H(0)^* H(0),$$

which gives from (53b)

$$\bar{P}_1(0)^* P_1(0) \leqq P_1(0)^* P_1(0). \tag{54}$$

Thus $P_1(z)$ is row outer.

Proof of 5) Suppose $H(z)$ is row outer. Then using the argument above, we can take $H(z) = [P_1(z), P_2(z)]$, where $P_1(z)$ is row outer. Put

$$G(z) = \begin{bmatrix} P_1(z)^{-1} \\ \\ 0 \end{bmatrix}.$$

Then by 2) of Theorem 10, $G(z)$ belongs to class N^+. It is clear $H(z)G(z) = I$. Conversely, suppose that there exists a matrix function $G \in N^+$ such that $H(z)G(z) = I$. Let $H(z) = H_i(z)H_o(z)$ be the inner-outer factorization of $H(z)$, where $H_i(z)$ is inner and $H_o(z)$ is row outer. Then we have

$$\det H_i(z) \cdot \det[H_o(z)G(z)] = 1.$$

Since $H_o(z)G(z)$ belongs to class N^+, we have the inner-outer factorization of $\det[H_o(z)G(z)]$,

$$\det[H_o(z)G(z)] = f_i(z)f_o(z),$$

where $f_i(z)$ is scalar inner and $f_o(z)$ is scalar outer. Hence

$$\det H_i(z) \cdot f_i(z)f_o(z) = 1,$$

which implies $|\det H_i(z)| = 1$. Thus $H_i(z)$ becomes a constant unitary matrix, which means that $H(z)$ is row outer.

<u>Proof of 6)</u> Every minor of order r in the matrix $H_i(z)H_o(z)$ possesses $\det H_i(z)$ as an inner factor by means of the Binet-Cauchy theorem [14, p.9]. Hence $\det H_i(z)$ devides all the minors of order r in $H(z)$. Since $H_o(z)$ is row outer, from 4) of Theorem 10, there exists an $r \times r$ submatrix $P_o(z)$ of $H_o(z)$ such that $P_o(z)$ is row outer. Let $P(z)$ be the corresponding square submatrix of $H(z)$ to $P_o(z)$ such that $P(z) = H_i(z)P_o(z)$. Then it holds $\det P(z) = \det H_i(z) \cdot \det P_o(z)$. Since $P_o(z)$ is outer, $\det P_o(z)$ is outer from 1) of Theorem 10. Thus $\det H_i(z)$ becomes a greatest inner divisor of $\det P(z)$. Hence $\det H_i(z)$ is a greatest common inner divisor of all the minors

of order r of H(z). Q.E.D.

A transfer function H(z) of bounded type with $H(e^{j\omega}) \in L^1$ is said to be causal if, in the Fourier series of $H(e^{j\omega})$

$$H(e^{j\omega}) \sim \sum H_n e^{j\omega n}, \tag{55}$$

the nth Fourier coefficient H_n vanishes for any $n < 0$.

The following result specifies the condition of causality of linear systems with transfer functions of bounded type.

Theorem 11: Suppose H(z) is an $r \times m$ matrix function of bounded type and $H(e^{j\omega}) \in L^1$. Then H(z) is causal if and only if H(z) belongs to class H^1. Moreover, suppose H(z) is represented by an irreducible MFD $H(z) = D(z)^{-1} N(z)$. Then H(z) is causal if and only if D(z) is outer.

Proof: Suppose the nth Fourier coefficient H_n vanishes for $n < 0$. By the Riemann-Lebesgue lemma [2. p.109], the nth Fourier coefficient $H_n \to 0$, as $n \to \infty$. This means from the Cauchy-Hadamard formula that the sires $\sum_{n=0}^{\infty} H_n z^n$ converges on $|z| < 1$. Thus we can define

$$H_+(z) = \sum_{n=0}^{\infty} H_n z^n. \qquad |z| < 1 \tag{56}$$

Since

$$H_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{-j\omega n} d\omega \qquad \text{for all } n, \tag{57}$$

it follows from (56) that

$$\begin{aligned} H_+(z) &= \sum_{n=0}^{\infty} H_n z^n + \sum_{n=1}^{\infty} H_{-n} z^{-n} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{j\omega} - z|^2} H(e^{j\omega}) d\omega, \end{aligned} \tag{58}$$

which is the Poisson integral representation of $H_+(z)$. This implies together with $H(e^{j\omega}) \varepsilon L^1$ that $H_+(z)$ belongs to class H^1 and that

$$H_+(e^{j\omega}) = \lim_{r\to 1} H_+(re^{j\omega}) = H(e^{j\omega}) \quad \text{a.e.} \tag{59}$$

(see [2, p.262] or [8, p.114]).

Since $H(z)$ is of bounded type, it follows from (59) that

$$H(z) = H_+(z), \qquad |z| < 1, \tag{60}$$

because functions of bounded type are uniquely determined by their boundary values a.e. on $|e^{j\omega}| = 1$. Thus $H(z)$ belongs to class H^+.

Conversely, suppose that $H(z)$ is of class H^1. Then we know that $H(re^{j\omega}) \to H(e^{j\omega})$ in the L^1-topology as $r \to 1$, that is,

$$\lim_{r\to 1} \int_{-\pi}^{\pi} |H_{ij}(re^{j\omega}) - H_{ij}(e^{j\omega})| d\omega = 0 \quad \text{for all } i, j, \tag{61}$$

where $H_{ij}(z)$ is the (i, j)th entry of $H(z)$ [2, p.368]. Since $H(z)$ is analytic in $|z| < 1$, the $\underline{n}$th Fourier coefficient of $H(re^{j\omega})$ vanishes for $n < 1$. This means together with (61) that the same is true for $H(e^{j\omega})$. Thus $H(z)$ is causal.

Suppose $H(z)$ is represented by an irreducible MFD $H(z) = D^{-1}(z)N(z)$. Then it is clear from Theorem 8 and (21'a) that $H(z)$ is causal if and only if $D(z)$ is outer. Q.E.D.

Some remarks will be given here. Suppose that $H(z)$ is rational and is represented by an irreducible MFD $H(z) = D(z)^{-1}N(z)$, where $D(z)$ and $L(z)$ are LIC polynomial matrices. Then $H(z)$ is causal if and only if $D(z)$ has no zero in $|z| < 1$. We can see from Theorem 11 that this statement is not correct in the case when $H(z)$ is irrational. In order to see actually this fact, consider the scalar transfer function

$$H(z) = D(z)^{-1}N(z),$$

where

$$D(z) = \exp\{\frac{z+1}{z-1}\} \quad \text{and} \quad N(z) = 1,$$

By an elementary calculation, we see that $D(z)$ is bounded in $|z| < 1$ and has no zero in $|z| < 1$. On the other hand, it holds

$$1/D(e^{j\omega}) = D(e^{-j\omega}) \quad \text{a.e.},$$

which implies that the nth Fourier coefficient H_n of $H(e^{j\omega})$ vanishes for $n > 0$, because

$$H_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega n}/D(e^{j\omega})d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega n}\, D(e^{-j\omega})d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega n}\, D(e^{j\omega})d\omega = D_{-n} = 0, \quad \text{for} \quad n > 0,$$

where D_n is the nth Fourier coefficient. Thus $H(z)$ is not causal.

IV. MATRIX-FRACTION DESCRIPTIONS FOR MATRIX FUNCTIONS OF BOUNDED TYPE

The matrix-fraction descriptions (MFDs) have been well developed for rational matrix transfer functions [12, Ch.6]. We shall extend the notion of the MFDs to matrix transfer functions of bounded type, and obtain some corresponding results to those in the rational case.

In Section III we obtained a left MFD (18) from a given $H(z)$ expressed by (17). We also have from (18) a right MFD

$$H(z) = N_R(z)^{-1}D_R(z), \tag{62a}$$

$$D_R(z) = d(z)I_m, \quad \text{and } N_R(z) = N(z) \tag{62b}$$

The following three theorems are basic results for the MFDs, and derived by Theorems 7, 8 and 10.

Theorem 12: Suppose that a transfer function $H(z)$ of bounded type has two irreducible left MFDs

$$H(z) = D_1(z)^{-1}N_1(z) = D_2(z)^{-1}N_2(z) \tag{63}$$

where $D_i \in H^\infty$ and $N_i \in H^\infty$ for $i = 1, 2$. Then there exists an outer function $F(z)$ of class N such that

$$D_1(z) = F(z)D_2(z), \quad N_1(z) = F(z)N_2(z). \tag{64}$$

Proof: It follows from (63) that

$$N_1(z) = D_1(z)D_2(z)^{-1}N_2(z). \tag{65}$$

Putting $F(z) = D_1(z)D_2(z)^{-1}$, we obtain

$$[D_1(z), N_1(z)] = F(z)[D_2(z), N_2(z)]. \tag{66}$$

Since $[D_1(z), N_1(z)]$ is row outer and $[D_2(z), N_2(z)]$ is analytic in $|z| < 1$, it follows from (66) and 4) of Theorem 10 that det $F(z)$ has no zero in $|z| < 1$. Analogously, since $[D_2(z), N_2(z)]$ is row outer and $[D_1(z), N_1(z)]$ is analytic, det $F(z)$ has no pole in $|z| < 1$. Thus $F(z)$ belongs to class N by means of Theorem 1. Therefore it follows from (65) and 3) of Theorem 10 that $F(z)$ is row outer for class N. Q.E.D.

Theorem 13: Suppose that a transfer function $H(z)$ of bounded type has two left MFDs

$$H(z) = D_1(z)^{-1}N_1(z) = D_2(z)^{-1}N_2(z) \tag{67}$$

where $D_i \in H^\infty$ and $N_i \in H^\infty$ for $i = 1, 2$. If $D_2(z)$ and $N_2(z)$ are LIC, then there exists a matrix function $L(z)$ of class N^+ such that

$$D_1(z) = L(z)D_2(z), \quad N_1(z) = L(z)N_2(z). \tag{68}$$

Proof: If $D_1(z)$ and $N_1(z)$ are LIC, Theorem 13 is true from Theorem 12. Supppose that $D_1(z)$ and $L_1(z)$ are not LIC, and let $U(z)$ be their GCLID, so that

$$D_1(z) = U(z)\bar{D}(z) \quad \text{and} \quad N_1(z) = U(z)\bar{N}(z), \tag{69}$$

where $U(z)$ is inner and $\bar{D}(z)$ and $\bar{L}(z)$ are LIC. Since

$$H(z) = \bar{D}(z)^{-1}\bar{N}(z) = D_2(z)^{-1}N_2(z).$$

it follows from Theorem 12 that there exists an outer function $F(z)$ of class N such that

$$\bar{D}(z) = F(z)D_2(z) \quad \text{and} \quad \bar{N}(z) = F(z)N_2(z). \tag{70}$$

Thus putting $L(z) = U(z)F(z)$, we see from Theorem 8 that $L(z)$ belongs to class N^+. Therefore (68) follows from (69) and (70). Q.E.D.

Theorem 14: Given a left and a right irreducible MFDs

$$H(z) = D_L(z)^{-1}N_L(z) = N_R(z)D_R(z)^{-1}, \tag{71}$$

a greatest inner factor of det $D_L(z)$ becomes a greatest inner factor of det $D_R(z)$.

<u>Proof:</u> By Theorem 13 together with (18) and (71), we have

$$[d(z)I_r,\ N(z)] = L(z)[D_L(z),\ N_L(z)], \tag{72a}$$

$$L(z) = U(z)F(z), \tag{72b}$$

where U(z) is inner and F(z) is outer. Analogously we obtain from (62) and (71)

$$\begin{bmatrix} d(z)I_m \\ N(z) \end{bmatrix} = \begin{bmatrix} D_R(z) \\ N_R(z) \end{bmatrix} \bar{L}(z), \tag{73a}$$

$$\bar{L}(z) = \bar{F}(z)\bar{U}(z), \tag{73b}$$

where $\bar{F}(z)$ is outer and $\bar{U}(z)$ is inner. Since $[D_L(z),\ N_L(z)]$ is row outer and F(z) is row outer, it follows from 3) of Theorem 10 that $F(z)\ [D_L(z),\ N_L(z)]$ becomes row outer. Denote

$$G(z) = [d(z)I_r,\ N(z)], \quad G_o(z) = F(z)[D_L(z),\ N_L(z)],$$
$$\bar{G}(z) = \begin{bmatrix} d(z)I_m \\ N(z) \end{bmatrix} \quad \text{and} \quad \bar{G}_o(z) = \begin{bmatrix} D_R(z) \\ N_R(z) \end{bmatrix} \bar{F}(z). \tag{74}$$

Then we have

$$G(z) = U(z)G_o(z) \quad \text{and} \quad \bar{G}(z) = \bar{G}_o(z)\bar{U}(z), \tag{75}$$

which are the inner-outer factorization of G(z) and $\bar{G}(z)$, respectively. Thus we know from 6) of Theorem 10 that det U(z) is a greatest common inner divisor of all the minors of order r in G(z), and that det $\bar{U}(z)$ is a greatest common inner divisor of all the minors of order m in $\bar{G}(z)$. But they are in fact greatest common inner divisors of the following collections: Suppose for definiteness $r \geqq m$. For the matrix $G(z) = [d(z)I_r, N(z)]$,

$$[d(z)]^r,$$

$$[d(z)]^{r-1}\cdot[\text{Minors of order 1 in } N(z)],$$

$$\vdots$$

$$[d(z)]^{r-m}\cdot[\text{Minors of order m in } N(z)].$$

For the matrix $G(z)^T = [d(z)I_m, N(z)^T]$ (where superscript T denotes transpose),

$$[d(z)]^m,$$

$$[d(z)]^{m-1}\cdot[\text{Minors of order 1 in } N(z)],$$

$$\vdots$$

$$[d(z)]^0\cdot[\text{Minors of order m in } N(z)].$$

Hence we have

$$\det U(z) = \det \bar{U}(z)\cdot[\text{a greatest inner factor of } d(z)]^{r-m}. \qquad (76)$$

On the other hand, we get from (72) and (73)

$$\det D_L(z)\cdot\det U(z)\cdot\det F(z) = [d(z)]^r, \qquad (77a)$$

$$\det D_R(z)\cdot\det \bar{U}(z)\cdot\det \bar{F}(z) = [d(z)]^m. \qquad (77b)$$

which implies

$$[\text{a greatest inner factor of } \det D_L(z)]\cdot\det U(z)$$

$$= [\text{a greatest inner factor of } d(z)]^r, \qquad (78a)$$

$$[\text{a greatest inner factor of } \det D_R(z)]\cdot\det \bar{U}(z)$$

$$= [\text{a greatest inner factor of } d(z)]^m, \qquad (78b)$$

because det $U(z)$ and det $\bar{U}(z)$ are inner and det $F(z)$ and det $\bar{F}(z)$ are outer. Hence we obtain from (76) and (78) that a greatest inner factor of det $D_L(z)$ becomes a greatest inner factor of det $D_R(z)$. Q.E.D.

V. CONCLUSIONS

We have considered the linear discrete-time systems with matrix-valued transfer function of bounded type. The notion of outer functions has been extended to matrix-valued function of the Nevanlinna class N, and the canonical factorization theorem for matrix-valued functions of class N has been presented. This theorem gives minimum phase systems for these linear systems, and specifies the necessary and sufficient condition for the systems to be causal. The notion of the MFDs has been extended to these linear systems, and some properties of the MFDs have been presented.

We would expect further applications of the canonical factorization theorem. The one application, which is under our investigation, is to providing a necessary and sufficient condition of the identifiability of linear systems operating in closed loop.

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